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Establish Shared Visions and Support Productive Adaptations on All Levels

Aims, Strategies, and Architecture of a Nationwide Implementation Program

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Abstract

Large-scale implementation programs for mathematics education innovations should combine several implementation strategies and span over several levels: the mathematics classroom level, the teacher professional development (PD) level, and the facilitator PD level, as well as over the systemic contexts on each of these levels. In this paper, we present the strategies and program architecture of the German nationwide 10-year implementation program QuaMath, which aims at developing the quality of mathematics classrooms and teacher PD in cooperation with the federal states, 400 PD

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facilitators, and, prospectively, 10000 schools. Given the spread of intended implementation and the depth of the targeted instructional innovations, we also outline the general philosophy of establishing shared visions with all stakeholders and support their productive adaptations at the same time. This philosophy also underpins the planned implementation design research in the program.

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Keywords

high-quality mathematics teaching – teacher professional development – facilitator professional development – implementation strategies – implementation architecture – shared vision – productive adaptations

1 Large-Scale Implementation Programs between Shared Visions and Openness to Adaptations

While there has been increasing consensus about what characterizes high-quality mathematics teaching (Hiebert & Grouws, 2007), school systems all over the world still work hard to develop strategies and structures to *implement* instructional approaches for high-quality teaching into large numbers of classrooms (Ahl et al., 2023; Coburn, 2003; Century & Cassata, 2016; Maaß et al., 2019; Koichu et al., 2021). A recent review on implementation research in mathematics education outlined that current publications have been dominated by qualitative case studies on teachers' reform enactment or effectiveness studies, whereas "planning and design aspects of such projects were rarely reported" (Ahl et al., 2023, p. 2). The intent of this paper is to report on the *planning and design aspects of a large implementation project* in Germany.

When developing implementation strategies and structures, two characteristics of successful implementation processes need to be coordinated: (1) *program coherence* in a shared vision of high-quality teaching (Cobb & Jackson, 2021) and (2) *openness* for context-specific adaptations (Morony, 2023). Whereas early conceptualizations of implementation tended to favor top-down approaches from research towards practice, evaluated in terms of implementation fidelity (as historically described in Century & Cassata, 2016), more recent conceptualizations have emphasized the needed shift in reform ownership (Coburn, 2003) and necessary *adaptation processes* by all stakeholders (Penuel & Fishman, 2012; Koichu et al., 2021). In particular, the implementation support should be "adaptable to different contexts and changing circumstances, accessible and sustainable" (Morony, 2023, p. 220). Such an openness for adaptations is easier to realize between innovation developers and stakeholders when a *limited* number of schools is addressed, but should also be taken into account for large numbers of schools (Penuel & Fishman, 2012). This openness seems to contradict the aforementioned characteristic of program coherence: For large-scale implementation projects of instructional innovations, a *shared vision* of high-quality teaching and a coherent set of principles for realizing this vision has been identified as essential (Newmann et al., 2001; Cobb & Jackson, 2021). In this paper, *we elaborate the argument that we have to and we can* coordinate both characteristics: *program coherence* through establishing shared visions of high-quality teaching and high-quality mathematics professional development (PD) and *openness for context-specific adaptations*. Furthermore, we argue that the shared vision needs to cover several levels.

We substantiate our argument by presenting the case of our ambitious 10-year nationwide implementation project in Germany that has been constructed based on both of these characteristics: the project QuaMath ("Developing *Qua*lity in *Math*ematics classrooms and teacher PD") is running from 2022 to 2033; is being carried by a consortium of 28 mathematics education professors, 40 academic project team members, 60 state project leaders, and 400 facilitators; and aims at reaching 10000 schools in 10 years, these are 30% of all primary and secondary schools in Germany. As an advance organizer, Figure 1 depicts the implementation structure by which we intend to reach 4000 schools in Phase 1 (2024–2028) and 6000 in the Phase 2 (2028–2033). Every year, 400 facilitators will work with 1000 new schools in 200 school networks. The first level concerns professional learning in the schools, the second level the teacher professional development with the school networks (TPD program), and the third level the facilitator professional development (FPD program).

In Section 2, we further explain the levels in connection with the aims of the program and sketch the (hopefully shared) coherent cores on each level. Section 3 substantiates the project insights with respect to the implementation architecture and the three implementation strategies. Both lay the base for Section 4, in which the main argument of needed support for productive adaptation is substantiated and research questions are derived for implementation-related design research on each level to reach this goal. Even though the 10-year project has just started, the presentation of its strategies, structures, and planned research may already be able to contribute to the

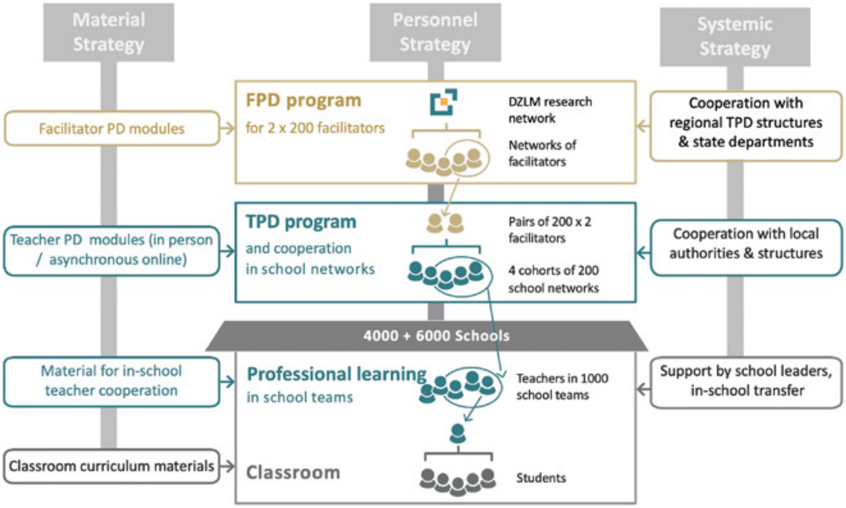

FIGURE 1 Implementation architecture of the QuaMath program with three implementation strategies on three levels (Prediger et al., 2024)

discourse on implementation research, even in countries beyond Germany (Ahl et al., 2023).

2 Aims and Coherent Cores of the QuaMath Program for Each Level

Figure 2 summarizes the multi-level aims of the QuaMath program that will be explained from bottom to top in the next subsections, closely following a project presentation originally published in German that will be elaborated in Sections 2 and 3 (Prediger et al., 2024).

2.1 *Aims and Coherent Core on the Student Level: Common Core Standards for Strengthening Competences and Reducing Disparities*

The long-term aim of the QuaMath program is to *strengthen students' mathematics competences* from Kindergarten to Grade 13. The competences in view are normatively prescribed in the common core standards of all federal states in Germany (KMK, 2022), with longitudinal coherence of fundamental ideas across all grade levels, and unpacked into the targeted conceptual understanding and procedural fluency for many mathematical topics and mathematical practices such as problem-solving, modelling, and reasoning across topics (similar to the CCSSI, 2010). To also address mathematical literacy aims beyond those measurable in large-scale assessments (Jablonka & Niss, 2014),

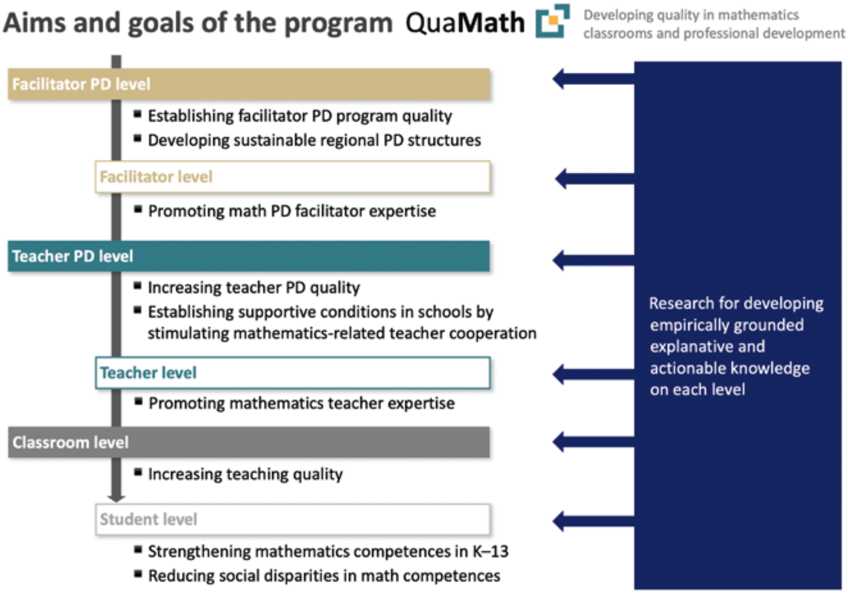

FIGURE 2 Aims and goals of the QuaMath program on six levels (including research aims for all levels)

the common core standards reveal an adequate curricular framework for program goals on the student level, with longitudinal progressions of mathematical practices and topics (KMK, 2022).

Like in other countries, the German school system bears huge gaps between the written curriculum established in 2004 and the achieved curriculum: In the last national survey, only 55% of fourth graders met or exceeded the "regular standards" (prescribing what a medium-successful student should achieve) and 22% even failed to meet the "minimum standards" (prescribing the minimum to continue in further schooling; Schumann & Sachse, 2022, p. 70). Among ninth graders, only a minority of 45% met or exceeded the regular standards and 24% failed to meet the minimum standards (Stanat et al., 2019).

The second aim of *reducing social disparities in mathematics competences* responds to repeated findings of international comparative studies that the German school system has been less able than other countries to compensate for social and immigration-related disadvantages (OECD, 2016). This second aim is to be pursued by a general increase in teaching quality for *all students* (DIME, 2007) and targeted approaches such as increasing agency for marginalized groups (Boaler, 2002), conceptually focused remediation approaches (Prediger, 2022), and language-responsive mathematics instruction (Prediger, 2019).

2.2 *Aims and Coherent Core on the Classroom Level: QuaMath Principles of Mathematics Teaching for Increasing Teaching Quality*

In order to achieve the two aims on the student level in the long run, an *increase in teaching quality* is the central aim on the *classroom level*. Empirical studies have repeatedly revealed that deep structures of teaching quality have a far greater influence on students' competence development than surface structures or external school organization (Hiebert & Grouws, 2007; Kunter et al., 2013). Thus, deep subject-specific structures are the focus of instructional development efforts in QuaMath. For large-scale PD and implementation programs, a coherent framework on instructional quality has been shown to be crucial, according to which all participants develop and pursue a shared vision of high-quality teaching (Newmann et al., 2001; Schoenfeld, 2014; Cobb & Jackson, 2021). Given the necessary flexibility for context-specific adaptations, we emphasize the *coherent core* more than a coherent framework, which does not leave much room for adaptations.

Such a coherent core for teaching quality cannot be based solely on *normative perspectives* (e.g., reducing social disparities by compensating disadvantages of marginalized student groups). *Empirical perspectives* informed from empirical research on teaching quality have provided insights into quality dimensions of effective teaching, such as cognitive demands, student support, and classroom management (Praetorius et al., 2018). However, the methodological focus on short-term measurable learning gains underlying many empirical studies might miss longitudinal connections along the educational chain (Schoenfeld, 2014). This is why the coherent core must also be grounded by *epistemological perspectives* (e.g., on long-term curriculum trajectories; see Bruner, 1966; Wittmann, 1998) and *pragmatic perspectives* about which principles of high-quality teaching can most productively guide teachers' actions (Schoenfeld, 2014; Prediger et al., 2022a).

Taking normative, epistemological, empirical, and pragmatic perspectives into account, a set of *five principles of high-quality mathematics teaching* was therefore specified in the DZLM (Deutsches Zentrum für Lehrkräftebildung Mathematik (German Center for Mathematics Teacher Education, a network of 12 German universities attached to the IPN Leibniz-Institute for Science and Mathematics Education)). In explicit connection to Schoenfeld's (2014) TRU-framework, but adapted to the German context, we specified the following five QuaMath principles for high-quality mathematics teaching (Prediger et al., 2022a; Holzäpfel et al., 2024):

- Cognitive demand: Initiate active learning processes
- Conceptual focus: Build a fundament for concepts, strategies, and procedures
- Longitudinal coherence: Prepare for sustainable learning

- Student focus and adaptivity: Work with students' perspectives
- Enhanced communication: Talk about mathematical ideas

The five QuaMath principles were discussed with all participating PD module designers (from the DZLM network and two further universities; see Figure 6 below for names) and state QuaMath leaders (from state PD institutes and state educational departments) and tested with PD facilitators for their potential to guide teachers' actions. Thus, the core of the *QuaMath vision of high-quality teaching* was finally established, which is coherently addressed in all PD modules. The resulting five QuaMath principles were elaborated theoretically and justified empirically in more depth in a conceptual paper (Prediger et al., 2022a) and presented for practical purposes in a professional journal paper (Holzäpfel et al., 2024). While various further principles could have been added (e.g., on sense making), the coherent use of a small core of only five shared principles has already proven helpful for the PD module design. In different PD modules, they are activated with flexible focal points and references to counteract the often-problematized fragmentation of German PD programs (Priebe et al., 2022).

2.3 *Aims and Coherent Core on the Teacher Level: QuaMath Framework for Promoting Teacher Expertise*

To achieve the aim of increasing teaching quality in the long run by processes of teaching quality development, the crucial goal on the teacher level is *promoting mathematics teacher didactic expertise* (Figure 3). The goal has been empirically justified by repeated findings that even with thoroughly designed curriculum materials, the didactic expertise of teachers is crucial for the successful implementation of teaching approaches (Brophy, 2000; Hill et al., 2005; Kunter et al., 2013; Depaepe et al., 2013).

QuaMath adopts a situated perspective on teacher instructional expertise (Putnam & Borko, 2000; Bruns et al., 2017) in which expertise is characterized by the teachers' practices for coping with typical instructional demands ("teacher jobs") of mathematics teaching (Bass & Ball, 2004; Prediger, 2019). To implement these practices in the classroom, they draw upon *content knowledge* elements (CK-C, on the classroom level) and upon *pedagogical content knowledge* elements (PCK-C) as their categories of perceiving and thinking (Bromme, 1992; Bass & Ball, 2004; Prediger, 2019; Gasteiger & Benz, 2018) or their interpretations of didactic principles as their action-guiding professional orientations (Schoenfeld, 2010; Prediger, 2019).

QuaMath coherently focuses on five instructional demands for which different high-leverage practices (Ball & Forzani, 2009) are addressed with regard to the five principles. In this way, a coherent 5 × 5 matrix is spanned across all PD

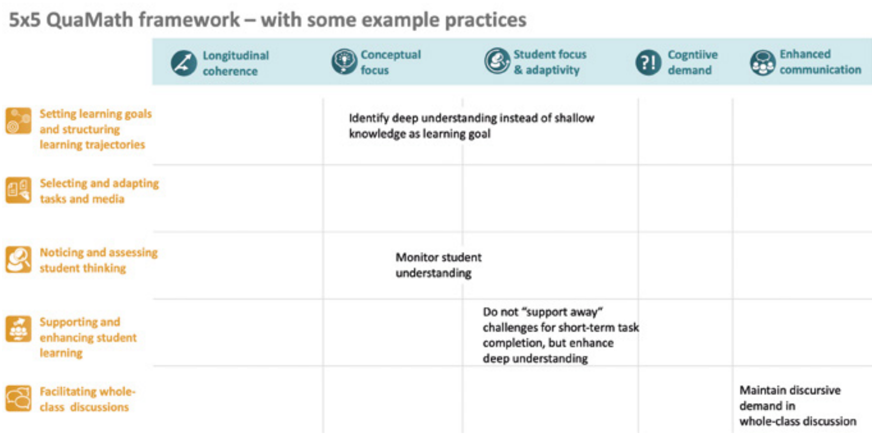

FIGURE 3 Shared 5 × 5 QuaMath framework for characterizing and addressing teachers' expertise as PD content, with example practices from the Mastering Math project (Prediger et al., 2022b)

modules into what we call the coherent "5 × 5 QuaMath framework." Figure 3 contains not only the framework, but also four example practices that have proven to be relevant for effective teaching or reducing disparities in a previous project, Mastering Math (Prediger, 2022). For example, a conceptual focus calls for particular practices of monitoring student thinking, namely those that focus on monitoring students' deep conceptual understanding rather than only shallow knowledge, so these must be set as learning goals before monitoring. Enhancing deep understanding (for a particular mathematical content area) is then also the relevant category guiding teachers' practices for enhancing student learning.

2.4 *Aims and Coherent Core on the Teacher PD Level: QuaMath Principles for Increasing Teacher PD Quality*

In order to achieve the articulated goals on the classroom level and teacher level, the QuaMath program pursues the goal of increasing the teacher PD quality, which also includes focused suggestions for stimulating mathematics-related teacher cooperation in different areas of PD content. The QuaMath program therefore refers to seven PD principles, starting with the coherence principle (Newmann et al., 2001):

- Coherence, Promote growth through coherent quality vision
- Competence orientation, Focus on productive practices for typical instructional demands

- Case focus, Work on cases from classrooms and experiment with new practices
- Participant orientation, Work with participants' perspectives and needs
- Reflection stimulation, Reflect individually and together on instructional practices
- Experiment model, Connect inputs, teaching experiments, and reflections
- Collaboration stimulation, Stimulate mathematics-related teacher cooperation

The articulation of the following six principles for high-quality teacher PD in the DZLM (Barzel & Selter, 2015) draws on the empirical state of research on effectiveness of PD programs (Garet et al., 2001; Lipowsky & Rzejak, 2015; Timperley et al., 2007; Yoon et al., 2007), which has increasingly taken into account not only the external design qualities but also the content qualities of PD programs, targeting the deeper structures of PD design.

While the general effectiveness of these PD principles has already been proven in generic empirical research, their content-related realization, especially with regard to content quality, is an ongoing process that is by no means a pure application of the principles. In contrast, each area of PD content for each PD module requires further research to specify, in a theoretically sound and empirically grounded manner (Roesken-Winter et al., 2021),

- which aspects of a typical instructional demand are productively dealt with using which high-leverage practices (competence orientation),
- which cases (e.g., excerpts from classroom teaching and learning) best carry the high-leverage practices and their backgrounds (case focus),
- what practices, categories and orientations teachers already bring to the instructional demands and into the PD and how these can be most appropriately leveraged to the intended high-leverage practices (participant orientation),
- what exactly needs to be reflected on in the context (reflection stimulation), and
- which content-related stimulations can deepen teachers' cooperation (collaboration stimulation).

All of these specification questions are addressed in iterative design research cycles at the teacher PD level (Prediger, 2019), which are pursued across several cohorts of teachers. In this way, the vision on high-quality teacher PD adopted across all PD modules can be substantiated for different PD modules with different PD content.

While international research has shown that *teacher cooperation* in professional learning communities can form a very beneficial condition for successful professional development (Borko & Potari, 2024; Wang et al., 2021), establishing subject-related teacher cooperation has not yet been sufficiently established in many German schools (Richter & Pant, 2016), so it is the most challenging principle to realize in our context. Given this lack in current school cultures and structures, we decided to outline collaboration as a challenging goal for teachers themselves on the teacher PD level. It is necessary to establish supportive conditions at schools through sensitizing school leaders (principals and lead mathematics teachers, if existent) for their (empirically found) central role in initiating and maintaining professional learning and quality development within their schools (Leithwood et al., 2020). The project is therefore aimed at establishing a culture of mathematics-related teacher cooperation, which will be new for Germany, by setting up QuaMath school teams for internal school cooperation and school networks for cooperation between 5 to 10 schools each. Materials for subject-related collaboration will be offered for each team that will be designed based on the state of research on professional learning communities (Borko & Potari, 2024; Selter & Bonsen, 2018; Roesken-Winter & Szczesny, 2016).

2.5 *Aims and Coherent Core on the Facilitator Level: QuaMath Framework for Promoting Facilitator Expertise*

To establish high-quality teacher PD, the PD approaches have been included in the PD materials and stimulations for teacher collaboration (the realization of which is described in Section 3). Similarly to classroom teaching quality, which substantially depends on teacher expertise, however, the teacher PD quality substantially depends on PD facilitator expertise (Borko et al., 2014; Lesseig et al., 2017) and the conditions under which the facilitators work.

Thus, the goal of the QuaMath program on the facilitator level is to promote facilitator expertise, and this is pursued on the facilitator PD level by the goal of establishing systematic facilitator PD programs (including preparation programs and continuous support). The design of the facilitator PD program draws on a *framework of facilitator expertise* according to which facilitators need to cope with typical demands of planning, conducting, and reflecting on teacher PD sessions with various practices (see Figure 4). For these practices, they draw on categories from six areas of knowledge (Wilhelm et al., 2019), three of which are also held by teachers: general pedagogical knowledge for classrooms (GPK-C), content knowledge for classrooms (CK-C), and pedagogical content knowledge for classrooms (PCK-C; Shulman, 1986). Facilitators' content knowledge for teacher professional development (CK-PD) can comprise aspects from

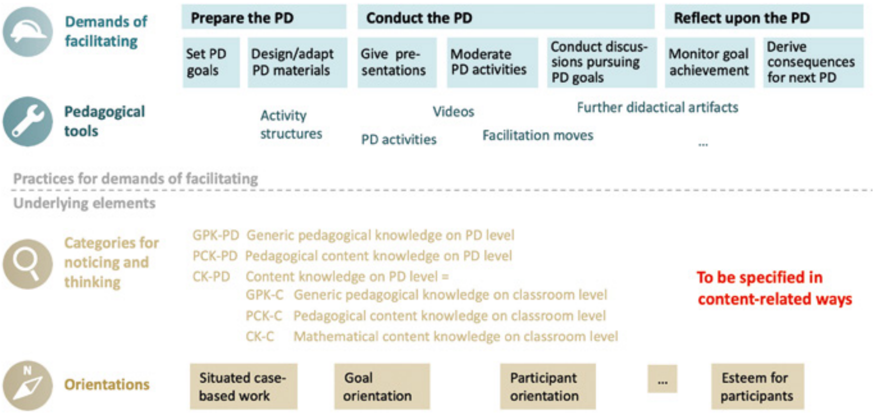

FIGURE 4 Coherent core on the facilitator level: Framework for facilitator expertise (Prediger et al., 2022b)

all three areas (CK-PD includes aspects of GPK-C, CK-C, and PCK-C), for which a deeper understanding is expected from facilitators than from teachers. In addition, facilitators need adult pedagogical knowledge (general pedagogical knowledge for teacher PD, GPK-PD), in other words, knowledge about generic pedagogies and activity structures for initiating collaboration in network meetings. Finally, the most important but often neglected area is *pedagogical content knowledge for teacher PD* (PCK-PD; Prediger et al., 2022b; Wilhelm et al., 2019), which can relate to aspects from all three areas of teacher knowledge, and refers to both PD content goals and teachers' typical starting points and challenges with respect to particular areas of PD content (e.g., How can teachers develop a more cognitively demanding use of digital media in arithmetic lessons? What categories should be offered to change teachers' perceptions of the arithmetic learning processes?). QuaMath places particular emphasis on a situated acquisition of PCK-PD categories (e.g., when analyzing teachers' contributions and moderating discussions) in the facilitator PD sessions.

Since facilitator PD programs for mathematics TPD courses have so far been offered relatively unsystematically in Germany (as problematized by KMK, 2020), the state education departments and the DZLM have jointly set a systemic policy goal for the QuaMath program: establishing sustainable regional PD structures together with the federal states with the aim of more systematically planning PD facilitator careers and developing the quality of facilitator programs.

2.6 *Aims and Coherent Core on the Facilitator PD Level: QuaMath Principles for Promoting Facilitator PD Quality*

With regard to the design of facilitator PD, the QuaMath principles for high-quality teacher PD (see Section 2.4) also apply here. In order to focus on the specific needs of facilitators, an eighth principle was added, which we will call the *consistent linking of classroom and TPD levels with deliberate reflection on changes in perspective*. The design principle of linking levels entails that the classroom level and the TPD level should be systematically connected in the facilitator PD. This can either be realized consecutively (e.g., 1st day classroom level, 2nd day TPD level) or more intertwined in shorter sequences (e.g., 15 minute classroom level, 15 minute TPD level). In both cases, facilitators can initially adopt teachers' perspectives and then explore teachers' professional learning processes at the TPD level, or vice versa, from the TPD activity to the classroom level.

By linking these levels, facilitators are not only enabled to act as teachers, but also to act as facilitators. In this context, it is important not to address aspects of GPK-PD and PCK-PD in isolation from each other, but always in relation to the subject matter of the PD program. This requires facilitators to constantly change their perspective, which should be practiced and consciously be implemented. It is important to always assume that teachers have comprehensive competences for professional activities at the classroom level. If this is not the case, it is necessary to supplement content at the PD level during qualification.

Overall, the QuaMath program thus aims to achieve goals at different levels for three groups of participants (students, mathematics teachers, and PD facilitators) using the DZLM implementation strategies in the three-tetrahedron model (Roesken-Winter et al., 2021; see Section 3.1). The QuaMath program provides a coherent core on each level to which all program components are aligned (Table 1). The program architecture and the implementation strategies are described in more detail in Section 3.

2.7 *Research Aims: Generating In-Depth Explanatory and Actionable Knowledge at All Levels*

Even though the program is grounded on the existing state of research, there is a considerable need for further research in order to generate empirically sound explanatory and actionable knowledge about the implementation processes and professional learning processes on the different levels (Prediger et al., 2019; Roesken-Winter et al., 2021). This includes not only retrospective evaluation and identification of obstacles, as in early implementation research (Century & Cassata, 2016; summarized in Ahl et al., 2023), but also design research for successful implementation (Penuel & Fishman, 2012; Cobb & Jackson, 2021),

TABLE 1 Intermediate summary: aims and coherent cores on six levels

Level	Aims and goals	Coherent cores as negotiated with small groups of stakeholders
Facilitator PD level	Establishing facilitator PD program quality	QuaMath principles of facilitator PD
Facilitator level	Promoting facilitator expertise	QuaMath framework of facilitator PD expertise
Teacher PD level	Increasing teacher PD quality	QuaMath principles of teacher PD
Teacher level	Promoting teacher expertise	QuaMath framework of teacher expertise
Classroom level	Increasing teaching quality	QuaMath principles of mathematics teaching
Student level	Strengthening mathematics competences	(National) Common core standards

which aims at theoretical contributions by generating explanatory and actionable knowledge, particularly in the following areas (Prediger et al., 2019; Roesken-Winter et al., 2021).

In iterative action research and design research approaches on the different levels, we qualitatively research a wide range of aspects (one content-related example is given in parentheses for each):

– Status and conditions of changeability of facilitator expertise with respect to practices relying on CK-PD and PCK-PD categories for different areas of PD content (e.g., within one module: How do facilitators' plan their TPD sessions, and what supports the depth of their participant-oriented and competence-oriented enactment?)
– Conditions of success and effects of selected design elements and content elements in facilitator PD modules (e.g., By which materials from TPD sessions can facilitators be engaged into deep reflections on teachers' orientations?)
– Status and conditions of changeability of mathematics teacher expertise (e.g., How do teachers select tasks with respect to low-achieving students, and how can we extend their ambitious vision for these students?)
– Conditions of success and effects of selected design elements and content elements in teacher PD modules and stimulations of teacher cooperation (including support measures for transfer within schools; e.g., How can video

cases support the reflection of teachers' practices to extend their ambitious visions for low-achieving students?)

We then combine it with later quantitative effectiveness research approaches in controlled trials using pre-post designs on:

– Effects of coordinated measures on several levels for growth of teacher expertise and teaching practices in classrooms (e.g., within one module: How have teachers' practices of setting learning goals and monitoring student understanding changed within half a year in the module?)
– Effects of coordinated measures on several levels for gains in students' mathematical competences (e.g., How have students' fraction competences increased with their teachers' participation in the PD module?).

In these different areas, we combine iterative action research and design research approaches with controlled trials, both mainly focused on particular modules. As we are convinced that PD research must take into account the PD content for each PD module, we conduct research in every of the 27 PD modules (see below for an overview on their contents). To this end, the iterative process of content-related design research on the PD levels (Prediger, 2019) is laid out over several cycles in the four cohorts, initially qualitatively formative, then quantitatively formative and quantitatively summative research. Due to the high level of complexity with many levels, it must be decided in each case in which areas pragmatic data collection and evaluation methods should be used and where the methodological rigidity for in-depth research should be applied, in other words, whether standardized measures are already available.

Finally, in Phase 2 (2028–2033), a more global evaluation of the medium-term effects will also be sought, and connections and cross-level effects will be investigated.

3 Implementation Architecture and Stakeholders in the QuaMath Program

In this section, we follow the call by Ahl et al. (2023) to outline "planning and design aspects" (p. 2) of the QuaMath program that have so far been "rarely reported" (Ahl et al., 2023, p. 2).

3.1 *Implementation Strategies and Implementation Architecture*

As early as 1994, Niss described the *implementation problem* as a key problem of mathematics education research to be treated when the role of mathematics in society should be improved and defined: The implementation problem deals with establishing the structural and organizational framework within which mathematics education is to take place. It further deals with providing the *immaterial resources* (e.g.,.... teaching methods, working forms), the *human resources* (teachers, consultants, mathematics educators) and the *material resources* (classrooms, textbooks, technology) for the realization of mathematics education. (p. 374, italics added)

Following the experiences from various other implementation programs and implementation research (Century & Cassata, 2016; Cobb & Jackson, 2021; Koichu et al., 2021; Penuel & Fishman, 2012; Wang et al., 2021) we developed and sharpened three implementation strategies that can be enacted on the classroom level, the teacher PD level, and the facilitator PD level (Roesken-Winter et al., 2021), with a focus on Niss's (1994) different resources:

- In material implementation strategies, we provide material resources to support actors on different levels, which means classroom curriculum materials for teachers, teacher PD materials for facilitators, and facilitator PD materials for facilitator educators.
- In personnel implementation strategies, we address human resources by providing professional development opportunities for the persons involved (teachers, facilitators, and state coordinators).
- In systemic implementation strategies, we address a critical part of Niss's (1994) immaterial resources, namely the systemic conditions under which the involved persons work. We try to take systemic hindrances into account and aim at partially improving systemic conditions, e.g., by establishing structures for networking and for exchanging of ideas. At the classroom level, this refers to teacher cooperation and school leader support in individual schools. At the teacher PD level, networks between schools and networks of facilitators are established, and at the highest level, we also establish a close cooperation between the PD structures of federal states and the QuaMath steering institutions (see next subsection).

Figure 1 has already shown the implementation architecture by which the strategies and network structures are established in QuaMath; they are described in more detail in the next subsections.

3.2 *Involved People and Timetable on Different Levels*

Figure 1 shows the different stakeholders and involved actors on the various levels. Behind the small DZLM symbol on the facilitator PD level, there is a large group of PD designer researchers in the DZLM network, which is coordinated by the IPN department for implementation research. Twenty-eight professors from the DZLM network and other universities are involved in module design and research as QuaMath module leaders, as listed in Figure 6 below. Kultusministerkonferenz (KMK, Standing Conference of State Educational Departments) funds will be used to support the development and research of the modules by 40 academic project team members (PD designers with PD facilitation experience and postdocs and PhD students as further PD design researchers) and administrative/technical staff. Many universities have acquired further third-party funding for research into the modules, meaning that additional employees will be involved. The QuaMath leading team consists of six professors from the DZLM network (S. Prediger and C. Selter together with H. A. Pant, B. Roesken-Winter, L. Holzäpfel, and D. Götze), five mathematics education researchers and one educational policy expert.

In line with the systemic strategy at the policy level, above the facilitator PD level, the project is steered by a policy steering committee (with participation of state secretaries of the state departments of education) controlling the complex consultation processes with all participants. The steering for each state is in shared responsibility with the QuaMath state leaders in each department of education and QuaMath state coordinators for primary and secondary education (each with a half-time position). The division of responsibilities means that around 60 people are involved at the steering levels of all participating states. The timetable in Figure 5 shows that the work with QuaMath state leaders and state coordinators began in August 2022.

In Spring 2023, 400 QuaMath facilitators were recruited by the 30 state coordinators. According to the facilitators' self-reports in the first survey (still unpublished), about one third were complete newcomers, one third had long-term experience and one third had some PD experience but no systematic

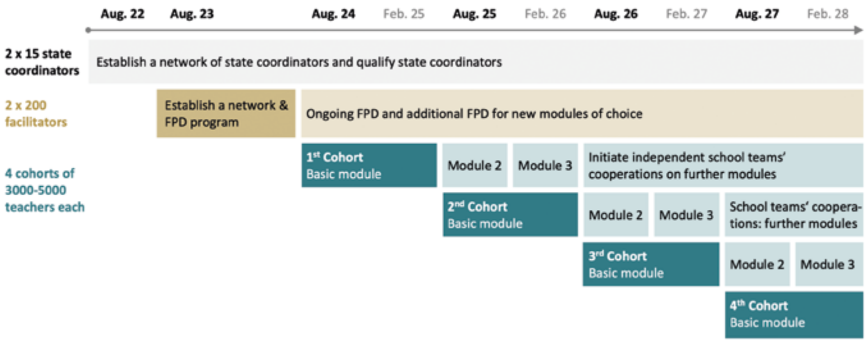

FIGURE 5 Timetable for various stakeholders and the first four cohorts of 1000 QuaMath schools each

PCK-PD learning opportunities. Those with experience had already worked in other state PD contexts. Some states transferred complete existing groups of facilitators to the new project, while other states recruited volunteers among their existing facilitators for the work in the new project. The facilitator PD program started in September 2023 and will be ongoing for the next 5 years. According to the contract with the states, the facilitators should all receive about 20% of work time for their project work (which has not yet been realized in all states).

The facilitators will later work in pairs, accompanying two networks of about five schools each. In a network of schools, three to five teachers from one school meet with a corresponding number of representatives from other schools to take part in regular network meetings. Schools in the various federal states were able to apply as volunteers to participate in the QuaMath program, with a joint conference decision and reliable project contracts. These procedures were initiated and controlled by the school administrations, with different recruitment processes across states.

In August 2024, the QuaMath facilitators will start with the first cohort of 3000–5000 teachers from 1000 QuaMath schools, who will be brought together in 200 networks. The aim of Phase 1 (2023–2028) is four cohorts, meaning a total of 4000 schools, and the aim of Phase 2 (2028–2033) is a further 6000 schools (not listed in Figure 5), so that after 10 years, the total will be 10 000 QuaMath schools at the primary and secondary levels. These 10 000 schools constitute 30% of the 33 000 schools in Germany.

3.3 *Experience from the Schools' Perspective*

3.3.1 Catalogue of Modules

As the timetable in Figure 5 shows, the work of a QuaMath school in a network generally spans at least 3 years. It begins in Year 1 with the Basic Module on high-quality mathematics teaching, which has been similarly designed for all primary and secondary schools. In Year 2, the state coordinators (possibly together with the facilitators) select two further modules to work on in the network of 5–10 schools. From Year 3 onwards, the schools can continue to work independently on asynchronous online modules of their own choice.

Figure 6 shows the catalogue of all 27 modules from which the state coordinators and facilitators can choose for Year 2 and the schools can choose from Year 3 onwards. The wide range of offered modules enables a needs-based focus that takes into account different school types and regional differences and the educational policy priorities of the different federal states.

School for Kindergarten Teachers	Primary Schools (Grades 1–4)		Lower Secondary Schools (Grades 5–10)		Higher Secondary (Grades 10–13)
Early Mathematics Gasteiger & Bruns	**Basic Module: High-Quality Mathematics Teaching 1–4** Selter & Götze		**Basic Module: High-Guality Mathematics Teaching 5–13** Holzäpfel & Prediger, Barzel & Greefrath (& Schacht)		
	Data & Measurement 1–4 Rösken-Winter	**Numbers & Operations 1** Bruns (& Gasteiger)	**Data & Probabilities 5–10** Rolka (& Rösken-Winter, Biehler)	**Functions & Modelling 7–10** Friesen (& Dreher, Rolka)	**Calculus & Derivatives** Schacht (& Barzel, Thurm, Greefrath)
	Geometry 1–4 Gasteiger (& Bruns)	**Numbers & Operations 2** Nührenbörger	**Algebra & Modelling 6–9** Dreher (& Friesen)	**Fractions, Percents, & Proportionals 6–7** Prediger & Friedrich	**Linear Algebra & Analytic Geometry** Wessel (& Kempen)
	Differentiated Instruction 1–4 Scherer	**Numbers & Operations 3–4** Scherer & Nührenbörger	**Assessing & Enhancing Arithmetic Concepts 5** Prediger (& Friedrich)	**Geometry 5–10** Kortenkamp	**Stochastics** Kempen (& Biehler, Wessel)
	Assessing & Enhancing Mathematics 1–4 Häsel-Weide	**Digital Media 1–4** Walter (& Selter)	**Digital Media 5–10** Kortenkamp (& Leuders)	**Language-Responsive Math Teaching 5–10** Prediger (& Wessel)	**Calculus & Integrals** Hußmann
	Language-Responsive Math Teaching 1–4 Götze	**Mathematical Practices 1–4** Höveler	**Problem Solving 5–10** Rott (& Holzäpfel)	**Differentiated Instruction 5–10** Friedrich (& Prediger)	

FIGURE 6 Overview of all QuaMath modules and QuaMath module leaders

3.3.2 Work in the Teacher PD Modules of the School Networks with Stimulation of Accompanied Teacher Cooperation

In each 6-month module, about five schools meet in three network meetings of 3 hours each (mostly in person, but also online if necessary in rural areas) with teaching experiments in between, all following an experiment model (Hiebert et al., 2003) that combines input, teaching experiments, and reflection phases.

The enactment of all PD modules in school networks by QuaMath facilitators is supported by thoroughly designed PD materials that are coherently aligned with the 5 × 5 QuaMath framework (Figure 3): They refer to the five QuaMath principles (shown in the column headings in Figure 3) in all modules and offer professional learning and reflection opportunities for all five instructional demands (shown in the row headings in Figure 3).

In the network meetings, the instructional demands and principles (from the 5 × 5 QuaMath framework) are thematized in approximations of practice (Grossman et al., 2009) and substantiated by offering suitable backgrounds. To this end, the facilitators receive videos and worksheets for PD activities and slide sets with inputs that can adopted and adapted for the network meetings, with a manual explaining the PCK-PD backgrounds and how they align to the PD principles. In order to compensate for the heterogeneous prior knowledge of participating teachers, additional materials (e.g., brief videos) are offered to work through background information individually.

The meta-analysis by Joyce and Showers (2002) reveals that outcomes of teacher PD programs greatly depend on the enacted PD components: Whereas theory input and discussion alone can have an impact mainly on teachers' abstract knowledge but little on their classroom practices, approximations of practice and feedback in the PD sessions can help to also promote teachers' skills. However, only teaching experiments (most preferably with individual coaching) can have an impact on the classroom practices teachers really enact. Given that the German PD structures do not have sufficient staff resources for individual coaching, we focus on teaching experiments (Hiebert et al., 2003) and collective preparation and reflection in school teams and network meetings. Classroom curriculum materials are provided that teachers can adopt for their contexts.

To stimulate teacher cooperation within the school teams without facilitators' moderation, materials are provided that invite the involvement of further teachers in a school, for example, through cooperative teaching experiments or focus questions for joint reflections.

3.3.3 Independent Continuation of School Team Collaboration from Year 3: Encouraging Continued Mathematics-Related Teacher Cooperation and Experiments

Teacher cooperation is internationally regarded as a highly supportive systemic condition for continued professional learning (Borko & Potari, 2024); however, in Germany it has so far only been established at a few schools (Richter & Pant, 2016). Establishing the organizational structure of school teams alone is not sufficient: DZLM studies have shown that cooperation is significantly more effective in terms of learning if it is specifically supported by *content-specific stimulations for each PD content area* (Selter et al., 2015).

That is why (after the first 2 years of intensive work with accompanying facilitators), the school teams are invited to independently continue their collaboration in self-chosen online modules for asynchronous professional learning: From Year 3 onwards, teachers at QuaMath schools (together with their colleagues who have not attended the networks in Years 1 and 2) can maintain and deepen their cooperation through the offered modules for asynchronous individual or team study on their own initiative and guided by their PD content interests (choosing from three to nine remaining modules for their respective grade levels; see Figure 6).

The online modules cover the same areas of PD content as the face-to-face modules in Year 2 and further themes for teaching experiments. They are developed from the face-to-face modules, yet optimized for unaccompanied teacher cooperation from Year 3 onwards by also offering formats such as videos, screencasts, interactive reflection offerings, animations, and self-checks and by systematically supporting the exchange in the school teams with focus on essential aspects of the 5 × 5 QuaMath framework.

One goal of further design research is to determine both how the cooperative quality development within and across schools (organizationally and in terms of content) functions effectively and what kind of external (thematic or human) support is most helpful at which point. The aim is to develop research-based approaches and materials in order to keep the cooperation alive over a longer period of time. To this end, we are constantly reviewing which network constellations and network themes are proving to be actionable and productive. In the paradigm of design research (Cobb & Jackson, 2021; Penuel & Fishman, 2012), targeted design elements and conditions for success need to be identified so that productive working structures can be stimulated and maintained with reasonable effort from the participating teachers.

4 Supports for Shared Visions and Productive Adaptations on Each Level

In Sections 2 and 3, we presented the implementation architecture, the common aims and goals for each level and the coherent cores that have been negotiated with different stakeholder groups.

The high relevance of *coherent cores* has been outlined in the comprehensive meta-study by Fixsen et al. (2005) in which they summarized reviews on ample research about effective implementation processes across different areas (mental health, juvenile justice, education, social services, etc.). The main outcome is that systematic implementation approaches are essential to any scaled-up spread of evidence-based innovations. Discussing the balance between program fidelity and openness for adaptations, they articulated the question "What must be maintained in order to achieve fidelity and effectiveness at the consumer level?" (p. 25), and answered,

> The answer is that core components that have been demonstrated to account for positive changes.... must be maintained. The core components are, by definition, essential to achieving good outcomes.... at an implementation site ... understanding and adhering ... may allow for flexibility in form (e.g., processes and strategies) without sacrificing the function associated with the component. (p. 25)

In this quotation, the traditional term "fidelity" from early implementation research is still used (as historically described in Century & Cassata, 2016), but reinterpreted in terms of program integrity with flexibility, taking into account scaffolded adaptations (Hill et al., 2022). The term "consumer" refers to all end-users such as teachers and students in a chain of levels.

For the QuaMath project, we assume our *coherent cores* (presented in Section 2) can serve this purpose of core components that give freedom for flexible adaptations while maintaining integrity (*focused flexibility*). However, in this section we discuss how coherent cores developed in *small groups* of stakeholders must first become *shared visions* for large groups of actors in the scaled-up project (outlined as an essential condition of success by Cobb & Jackson, 2021), and how context-specific adaptations need to be supported (outlined as a critical condition by Hill et al., 2022; Morony, 2023; Wang et al., 2021). This applies to actors and adaptations on each level.

In the next subsections, we revisit the *coherent cores* we have articulated for our six levels, describe typical challenges in their appropriation as *shared visions* and give examples for the support needed for productive adaptations.

4.1 *Student Level: Shared Visions on Core Standards and Supported Adaptations of Classroom Materials*

On the student level, QuaMath refers to the *core curriculum framework* as articulated in the common state standards (KMK, 2022), on which all 16 federal states agreed and which were recently revised with expertise from schools, facilitators, educational leaders, and researchers from the DZLM research network. However, this agreement on the written curriculum and its official status does not automatically imply that all actors subscribe to this core as their *shared vision* for students' most essential learning goals: If, for example, assessment designers neglect certain competences in the high-stakes state assessments, then teachers will not consider these competences as relevant learning goals to strive for. Constructive alignment between the written curriculum, textbooks, and state assessments is thereby crucial for achieving program coherence in an implementation process (Cockcroft Report, 1982; Newmann et al., 2001). Additionally, even if the nationwide standards prescribe normative standards as obligatory learning goals, a federalist nation such as Germany must live with different prioritizations between learning goals in different states' syllabus and state-wide assessment traditions.

In our practical work with facilitators and teachers from different states, these heterogeneous prioritizations have immediate consequences for the selection of classroom materials: if some mainstream learning goals do not agree perfectly in both the state syllabus and a school's syllabus, teachers hesitate more to invest time in using the materials. Since state coordinators and facilitators decided to adapt the materials for a better fit to their state syllabus with heterogeneous success in keeping the main ideas, we learned to deepen the conversation. As a consequence, the module designers understood that the design (i.e., the selection) of the classroom materials must already offer a wider range of curriculum fits (Wang et al., 2021). Additionally, the materials must make more explicit what principles they intend to exemplify (Wittmann, 2021; see next subsection).

4.2 *Classroom Level: Shared Visions on High-Quality Teaching and Supported Adaptations of Teaching Experiments*

On the classroom level, the implementation process is substantially shaped by the fact that the *innovation in view* of QuaMath is not a small, well-delineated standardized intervention program for a single mathematical topic (content or practice), but a comprehensive instructional approach from kindergarten to Grade 13 across all areas of mathematical content and practices that is guided by a *coherent core of five principles of high-quality mathematics teaching* (Prediger et al., 2022a): cognitive demand, conceptual focus, longitudinal coherence, student focus and adaptivity, and enhanced communication.

These principles provide a coherent core that was negotiated with many stakeholder groups and experimented with in facilitator groups to determine whether they were helpful to leverage teachers' practices (Prediger et al., 2022a). Most state leaders were involved in that process, and most state coordinators have already appropriated the principles in several sessions of initiation work. Even if some teachers and facilitators were already involved in their articulation, we now face the challenge to invite 400 facilitators and 3000–5000 teachers per year to adopt them as a *project-wide shared vision*. This requires the participant-oriented work on valuing teachers' perspectives and leveraging them towards the five principles while leaving room for further individual or school-specific quality principles. To support the flexible but targeted appropriation, we developed 27 modules for different age levels and areas of PD content (see Figure 6) with a focus on the five principles and five instructional demands of teaching (5 × 5 QuaMath framework; Figure 3).

In order to invite teachers to engage in a shared vision, teaching experiments are crucial. In 1982, Wittmann had already promoted teaching units as an integrating core for didactic principles (Wittmann, 1982/2021) by which teachers can experience the combination and substantiation of principles in holistic and topic-specific ways. Later research on professional learning emphasized the need for teachers to adapt teaching units while keeping the core ideas stable, so that instructional designs of the material must be *robust* for different teaching contexts (Burkhardt, 2006). We would add that the preparation and reflection of the teaching experiment should not dive too much into technical details (e.g., of each task formulation) but explicitly connect the teachers' plans and experiences with the underlying ideas of the five principles. Adaptations that teachers conduct should be celebrated as incidences of agency and constructively discussed with respect to their productive or unproductive contributions to strengthen the realization of the principles.

4.3 *Teacher Level: Shared Visions on Teacher Expertise and Supported Adaptations of TPD Materials*

As a coherent core on the teacher level, *the model of teacher expertise* (Bromme, 1992; Prediger, 2019) was transferred by referring the five principles to five recurrent instructional demands in the 5 × 5 QuaMath framework (Figure 3). The situated treatment of the five principles within five recurrent instructional demands supports teachers' appropriation of a shared vision of the principles.

Establishing a shared vision on the principles and instructional demands is also demanded in another stakeholder group, the *module design research teams* with 28 professors (called module leaders) and their teams (PD facilitators, PhD students, docents, etc.). The module leaders were involved in the consensual articulation of the principles so that the shared vision was already carrying the collective project application. Nevertheless, the module design for many different areas of PD content and age groups (Figure 6) requires a highly flexible interpretation of the principles with age-specific and mathematical content-specific and PD content-specific substantiations in each module. For example, *enhancing communication* on numbers and operations in Grade 1 differs substantially from communication on stochastics in Grade 13, and in the module on digital media, it has a very particular focus differing from the module on differentiated instruction. However, there is still the common core of the twofold principle focusing on communicating to learn and learning to communicate.

The complete QuaMath team is in a constant negotiation and mutual consultancy process to find the balance between common core and necessary age-specific and content-specific adaptations for the principles and instructional demands on the classroom level. However, the pure articulation of five instructional demands helps us to avoid scopes that are too narrow for PD activities (e.g., only solving and analyzing tasks).

4.4 *Teacher PD Level: Shared Visions on PD Principles and Supported Adaptations of TPD Session Enactment by Facilitators*

To articulate a coherent core on the teacher PD level, the state of research on effective TPD programs was condensed into *six principles for high-quality teacher PD* (Barzel & Selter, 2015), negotiated with the educational authorities and experienced facilitators in the last 10 years and supplemented by the coherence principle (Section 2.4). However, this does not mean that all involved actors will immediately adopt them as a shared vision, and neither will all members of the module design teams (see last subsection; constant negotiations and mutual consultancy are required) nor the 400 facilitators. Therefore, the TPD modules are crucial to discuss the backgrounds and interpretations of the principles and their enactment in the facilitator network meetings.

Even if TPD materials are thoroughly designed with respect to the principles, they are not meant as scripts to be enacted word by word with fidelity by facilitators. Instead, we expect the facilitators to adapt them to their contexts, their participant groups, age levels, and regional classroom cultures. As not all adaptations are equally productive, we conducted qualitative adaptation studies to better understand facilitators' decision making (Leufer et al., 2019). This allowed us to learn how to make the underlying ideas and structures so explicit that the facilitators' adaptations can be productive.

The discussion of adaptation ideas has become a major part of the facilitator PD sessions. The TPD material was focused on three to five take-home messages for each TPD session. These take-home messages are intended to guide teachers' attention and to signal to the facilitators what is definitely at the center of the PD session and should not be omitted.

4.5 *Facilitator Level: Shared Visions on Facilitator Expertise and Supported Adaptations of Facilitator PD Materials*

On the facilitator level, the coherent core comprises the framework of facilitator expertise (with its particular focus on content-specific PCK-PD categories; Figure 4) and the program structures and responsibilities to develop sustainable regional PD structures.

When designing the facilitator PD materials, for example, the designers in QuaMath must take into account that facilitators with very different experiences and competences take part in the facilitator PD sessions. A differentiated offer is therefore made, in which certain elements are outsourced in advance in so-called mini-blocks so that missing knowledge can be addressed.

4.6 *Facilitator PD Level: Shared Visions on Principles of Facilitator PD and Supported Adaptations of Facilitator PD Trainings*

For articulating a coherent core on the facilitator PD level, the seven *principles for high-quality facilitator PD* were supplemented by the principle of linking levels (see Section 2.6). However, this does not mean that all involved members of the module design teams will adopt them as a shared vision. It is therefore crucial to discuss the backgrounds and interpretations of the principles and their enactment in the QuaMath FPD design team meetings.

Although the QuaMath state leaders and state coordinators share the principal vision of establishing facilitators as a profession (that is new in Germany), resource restrictions still challenge the process. For example, some states had such a teacher shortage that they felt not able to grant the facilitators the 20% working hours for the project. This kind of "adaptation" has already turned out to be unproductive, as an important condition is not fulfilled. Nevertheless, QuaMath will have to develop strategies for dealing with the different time quotas of the facilitators.

5 Conclusion

While many programs have outlined their shared visions on the classroom level (Wang et al., 2021; Cobb & Jackson, 2021; Schoenfeld, 2014), this paper emphasizes that shared visions need to be established with coherent levels on each program level: the teacher level, the teacher PD level, the facilitator level, and the facilitator PD level. Overall, the QuaMath program provides a coherent framework for a coherent core on each level to which all program components are aligned. In future papers, we will explore how these coherent cores are actualized in the teacher PD and the facilitator PD programs. The planned research will also allow us to account for situated effects and measurable effectiveness of different parts of the program that cannot yet be reported.

Even if coherent cores seem to be a necessary condition for the success of a large program (Newmann et al., 2001), it is by far not sufficient that the QuaMath leading team has codified these coherent cores (Prediger et al., 2024, Holzäpfel et al., 2024) after discussion with some selected stakeholders from each level. When scaling up programs, it is critical that all actors in the program adopt the coherent cores as their *shared vision* (Cobb & Jackson, 2021) and *adapt* them to their context-specific conditions. Teachers and facilitators are not the only actors: They also include module designers, state coordinators, facilitator educators, and policy makers. It is important for all levels to communicate that adaptations do not mean that anything goes, but that ownership grows with the coherent core.

As Table 2 summarizes, these different supports for shared cores and supported adaptations refer to both material and personnel strategies: On the student, teacher, and facilitator levels they refer to the *material strategy* of QuaMath, while the cores and adaptations with respect to the classroom, teacher PD, and facilitator PD levels address the *personnel strategy* (see Section 3).

TABLE 2 Overview: coherent cores and supported adaptations

Level	Aims and goals	Coherent and hopefully shared cores	Supported adaptations of …
Facilitator PD level	Establishing facilitator PD program quality	QuaMath principles of facilitator PD	FPD sessions enacted by facilitator educators
Facilitator level	Promoting facilitator expertise	QuaMath framework of facilitator PD expertise	FPD materials
Teacher PD level	Increasing teacher PD quality	QuaMath principles of teacher PD	TPD sessions enacted by facilitators
Teacher level	Promoting teacher expertise	QuaMath framework of teacher expertise	TPD materials
Classroom level	Increasing teaching quality	QuaMath principles of mathematics teaching	Teaching experiments enacted by teachers
Student level	Strengthening mathematics competences	(National) Common core standards	Classroom curriculum materials

As the large implementation project QuaMath only started running in 2022, we cannot yet show any measurable effects. However, we hope that reporting about its structures and characteristics, coherent cores, and supported adaptations can contribute to the emerging mathematics education discourse on implementation projects. We close this paper by summarizing it through the lens of the definition of implementation as suggested by Koichu et al. (2021):

> We conceptualize implementation in mathematics education as an ecological disruption to a particular mathematics education system, through the gradual endorsement of innovation in conjunction with an action plan aimed at resolving what is perceived as a problem by (at least some of) the stakeholders involved. The defining feature of implementation is that it occurs in interaction between the innovation and plan proponents

> and the innovation adapters. At the beginning of the implementation, the innovation proponents have the ultimate agency over the innovation and the associated action plan. During the implementation process, the innovation adapters experience some or all of the following sub-processes:
> (1) constructing agency over the innovation,
> (2) gradually changing within-community communication or across-community communication,
> (3) gradually changing practice so that it accommodates the innovation,
> (4) adapting the innovation to their needs and aspirations.
> These subprocesses reflect back on the proponents, including evolution of the innovation, of the associated action plan and of the theories underlying their development. (p. 986)

First of all, we emphasize that in the German culture, *too radical ecological disruptions* would not find any acceptance on any level. Thus, our aim is that these changes emerge gradually while consequently valuing and building upon the assets that all actors already bring in. This applies to teachers' familiar teaching practices on the classroom level, but also to the facilitators' practices on the teacher PD level or to policies about existing regional PD structures.

Complexity is endowed by the nature of the *innovation in view* of QuaMath, which is not a well-delineated standardized intervention program for a single mathematical topic (content or practice), but a comprehensive instructional approach from Kindergarten to Grade 13 across all areas of mathematical content and practices that is guided by a common core of five principles of high-quality teaching (see Section 2.2 and Figure 3).

One first step of (1) *constructing agency over the innovation* (Coburn, 2003) is that all actors (teachers, facilitators, state leaders, PD module designers, and educational authorities) need to gain agency over these five principles by appropriating them for their work and interpreting them within a shared vision, but with flexibility for different contexts. For these processes, we have developed structures in our implementation architecture on several levels (Figures 1 and 2) to stimulate reflection and cooperation in participant-oriented and competence-oriented manners.

While structures are relatively well established in Germany to (2) *augment across-school communication* in school networks of about five schools, it is very challenging to coordinate every states' policies (each claiming their federalist autonomy) and to *stimulate within-school communication* for which the German school culture does not have a strong tradition. For this, we have a strong design ambition to support the within-school cooperation and communication with thoroughly designed online modules and two preparatory years.

A critical point in (3) *gradually changing practices* is that the German school system currently has no capacity for individual coaching, a characteristic that has been outlined as highly efficient for changing practices (Joyce & Showers, 2002; Cobb & Jackson, 2021). For the facilitators' practices, the aim is to compensate for this by working in pairs so that they can coach each other. For the teachers' practices, we hope to at least partially compensate for the lack of coaching by well-prepared teaching experiments (with preparation and reflection stimulation including classroom videos) so that the school teams might adopt at least some functions of the missing individual coaching. We also hope to keep the school teams continuing their work even after 3 years of project adherence.

For (4) *adapting the innovation to their needs and aspirations*, we need to consider the different actors on all levels. As Section 4 outlined, this seems to be the most critical aspect to be supported when the innovation in view is not well delineated and local.

While the last sentence (5) of the quotation is not numbered, we strongly agree that all these subprocesses "reflect back on the proponents … and … the theories underlying their development" (Koichu et al., 2021, p. 986). In our case, this means that the communication processes productively challenge us to increasingly make explicit the theoretical assumptions and practical implications underlying the coherent cores on each level. For example, this consideration has already led us to reformulate the PD principles (from Barzel & Selter, 2015; Timperley et al., 2007) so that they better align to the classroom quality principles. The integrative theorizing is an ongoing process that will continue over the next years.

In summary, we are working on the experienced-based assumption that the balance between adherence to the program agenda and flexibility for context-specific adaptations can be best achieved by

- creating a coherent conceptual core on each level,
- establishing it as a shared vision of all actors, and
- supporting the processes of adaptations by initiating communication about it.

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